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JOINT TRANSLATION BY CANTILEVER MOMENT DISTRIBUTION

by L. E. Grinter, M. ASCE
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PAPERS

JOINT TRANSLATION BY CANTILEVER MOMENT DISTRIBUTION

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SYNOPSIS

The Hardy Cross method of moment distribution has solved the problem of joint rotation in continuous structures in a simple manner. Joint translation (or sidelurch), however, has had to be dealt with in an indirect manner. The methods of correcting for sidesway include (1) an independent distribution of moments corresponding to each degree of freedom for translation, with interrelationships established by the solution of a number of simultaneous equations equal to the number of the degrees of freedom, (2) a sequence of partial sidesways, with joints restrained against rotation, which converge to the actual sidesway and form a series of corrections to the process of moment distribution, and (3) a reversed technique by which the computed or estimated movements of the joints that accompany sidelurch are used to determine the fixed-end moments for distribution. The method last mentioned has been effective in solving the problem of wind-stress analysis for tall buildings.

The concept of cantilever moment distribution is presented in this paper as a direct method of permitting joint translation to take place as an integral part of the process of moment distribution. Three techniques are suggested for application of the method: (*a*) In many structures involving identical columns or chords, complete freedom of sidelurch is permitted by cantilever moment distribution without any special correction. (*b*) For structures with unlike columns or chords an approximate solution by cantilever moment distribution that does not eliminate all restraining forces at the joints can be used. (*c*) Cantilever moment distribution can also be used to determine the moments and lateral deflections of a substitute structure having identical columns or chords. Following this analysis, the lateral joint movements of the substitute structure may be used in the previously cited reversed technique to determine a relatively accurate set of moments caused by sidesway in the structure with unlike columns or chords.

In the usual process of moment distribution, the relaxation or rotation of the joints, when joint translation is restricted, permits the reduction or loss of shear in members influenced by sidesway. For example, the bent shown in Fig. 1(*a*) has fixed-end moments introduced into the columns as shown in Fig. 1(*b*), so that the sum of the column shears equals the applied load P. When the joints B and C, however, are permitted to rotate without translation, the process of moment distribution as developed by Hardy Cross, Hon. M. ASCE,[3] results in reduced column shears which no longer balance the load P. It is evident that a variation of this system of moment distribution would be useful for lessening or—preferably—preventing shear loss. In the Appendix an analysis of the method of moment distribution clarifies the fact that two factors defining the degree of fixity for each member, which are determinable from the actual or assumed conditions of end restraint, define a process of moment distribution. Of the infinite variety of distribution procedures possible, only one, termed "cantilever distribution," has the characteristic of allowing the member to undergo no change in shear as moment distribution occurs.

[3] "Analysis of Continuous Frames by Distributing Fixed-End Moments," by Hardy Cross, *Transactions*, ASCE, Vol. 96, 1932, pp. 1-10.

Following the preparation of this paper, the writers became aware of other investigations of the subject of translation as a part of the process of moment distribution.[4,5]

[4] "'Degree of Fixity' Methods for Certain Sway Problems," by R. H. Woods and E. Goodwin, *The Structural Engineer*, July, 1953, Vol. XXX, pp. 153-162.

[5] "Analysis and Design of Columns in Frames Subject to Translation," by T. C. Kavanagh, *Column Research Council Report*, 1950.

Cantilever Moment Distribution.—Fig. 1(*c*) shows the distortion of a symmetrical bent that moves freely under applied corner moments ΣM. Each column has a moment diagram which corresponds to a free cantilever, since identical columns have the same lateral deflections under equal applied moments. For such a cantilever the moment carry-over factor to the fixed-end is -1.0, by use of the rotational sign convention introduced by one of the writers.[6] In Fig. 2 the slope θ, caused by a moment M, is $\frac{M L}{E I}$ as contrasted to $\frac{M L}{4 E I}$ in Fig. 3. (L is the length of the column; I denotes the moment of inertia of the cross section of the column; and E is the modulus of elasticity of the column.) Hence, for cantilever moment distribution, the stiffness factor is only 25% of the stiffness factor K that is used for a member in which joint translation is prevented. The stiffness factor and carry-over factor for cantilever moment distribution therefore equal $K/4$ and -1.0, respectively. These values can be compared with a stiffness factor of K and a carry-over factor of $+0.5$ for moment distribution without joint translation.

[6] "Sign Conventions," by L. E. Grinter, *Engineering News-Record*, March 7, 1935, Vol. 114, p. 359.

Cantilever Distribution for a Simple Bent.—Fig. 4 shows a simple bent with fixed-end columns and all members having the same K-values. The top member, for anti-symmetrical moments caused by sidesway has a stiffness, or resistance to equal end rotations, that is 1.5 times greater than its resistance to rotation at one end when the other end is fixed. Its corrected stiffness, or

Kv-value is $1.5 \times 4 = 6$. The column as a cantilever has a Kv-value equal to $0.25 \times 4 = 1$. The fixed-end column moments caused by the load P are $+140$ at each end of each column. When a moment distribution is performed at joint C of Fig. 4, the unbalanced moment of $+140$ causes a correction moment of $-(1/7) \times 140 = -20$ in member CD at end C, and a correction moment of $-(6/7) \times 140 = -120$ in member CB at end C. The carry-over factor for cantilever moment from joint C to support D is -1, which explains the correction moment of $+20$ at support D. Anti-symmetry then results in the moments at support A and joint B. The final moments at points A and D are $+160$ and the corner moments at points B and C are $+120$. It can be seen that a single joint-moment distribution, using the cantilever concept for the columns, completed the analysis of this bent for sidesway.

Cantilever Distribution for a Multi-Story Bent.—In Fig. 5 a three-story bent is analyzed by cantilever distribution. There is anti-symmetry of the moment diagrams for the girders; therefore, their stiffnesses are increased 50% and the moments are distributed at the joints of one column only. Fixed-end moments are proportional to the story shears for equal story heights and are $+50$, $+100$, and $+150$, respectively, from the third to the first story. The first joint to be balanced is B, at which the sum of the fixed-end moments is $+250$. Each column entering joint B is considered to act as a cantilever with its free end at joint B and its far ends, A and C, fixed against rotation. To maintain rotational continuity at joint B, the total fixed-end moment is distributed in proportion to the Kv-values, and, to avoid a linear discontinuity at joint B, two of the three joints A, B, and C must provide freedom for lateral joint translation.

Considering the moment distribution at joint B, the division of moment according to the Kv-values is -50 to each column and -150 to the girder. Then $+50$ is the carry-over moment to joint A and to joint C. At joint C the unbalanced moment is $+200$ and its distribution, according to Kv-values, gives -40 to members CB and CD and -120 to the girder at joint C. The third joint to be balanced is D, after which the distribution sequence B-C-D is repeated twice to obtain the final convergence. When any joint is being balanced, the two columns entering the joint are dealt with as if they were cantilevers with fixed far ends that require freedom of sidesway at the joint being balanced and at the next joint above, to permit double cantilever deflection. All other joints beyond the next joint above the one being balanced translate the same distance as does the next joint above the one being balanced.

Parallel-Chord Vierendeel Truss, Upper and Lower Chords Having Equal Stiffnesses.—In Fig. 6 a parallel-chord Vierendeel truss is shown, with upper chord and lower chord of equal stiffnesses and loaded at the panel points only. Under the imposed loads, the joint A′ moves to a′ and joints B, C, D, and E deflect to b, c, d, and e, respectively. Neglecting direct-stress deformations, the deflected positions of the remainder of the joints are fixed by the geometry of the structure. If the entire truss is given a rigid-body rotation until member Aa′ is vertical, all web members will be vertical, and joints A and F will not be on the same level. Since the reactions at joints A and F are statically determinate, they are known and may be retained as forces acting on the rotated truss. The angle of rotation is so small that the directions of all forces

can be considered to remain unchanged. The five degrees of freedom (four vertical joint movements and one horizontal movement) consist of the five free vertical deflections of joints B, C, D, E, and F. The parallel-chord Vierendeel truss with chords of equal stiffnesses is therefore structurally identical to the single-bay multi-story bent, and the method of cantilever moment distribution can be used for its analysis. Cantilever moment distribution of the Vierendeel truss is performed in sequence at joints A to F, and repeated until convergence is obtained. The necessary joint translations occur as a part of the process of cantilever moment distribution. The complete analysis of a three-panel unsymmetrical Vierendeel truss with unsymmetrical loading is shown in Fig. 7. The only limitation is that the upper and lower chords in each panel shall be of the same stiffness. An approximate solution for a structure with chords of unequal stiffnesses can be obtained by a procedure which utilizes a substitute structure with identical chords.

Appendix. Moment Distribution and Boundary Conditions

The moment distributed to each member at a rigid joint depends entirely (1) on the product $E\,K$ of each member, and (2) the two properties of each member which together are called the "degree of fixity." Two linear, independent, homogeneous equations connecting pairs or groups of five elements $\left(M \text{ and } \theta \text{ at the near end and } M \text{ and } \theta \text{ at the far end, and the sidesway } \frac{\delta}{l}\right)$ or giving zero values to certain of these elements, are necessary and sufficient to define the degree of fixity of each member. For example, the degree of fixity of the members shown in Fig. 8 is defined by the following sets of equations:

For member OA

$$\delta_{OA} = 0 \qquad (1a)$$

$$\theta_A = 0 \qquad (1b)$$

and for member OB

$$M_{BO} = 0 \qquad (2a)$$

$$\frac{M_{BO} + M_{OB}}{l_{OB}} = -\,c\,\delta_{OB} \qquad (2b)$$

in which c is a spring constant or the upward vertical reaction at end B per unit settlement of B.

If the product $E\,K$ and the degree of fixity of each member are known, as in Fig. 8, it is possible to use the two slope-deflection equations for each member to determine the moments in terms of θ_0 as will subsequently be shown in Eqs. 5 and 6. The moment-distribution factors and the carry-over factors can then be computed and the final moments in each member can be obtained by a single cycle of moment distribution. Thus, at joint O (Fig. 8) the two slope-deflection equations for each member meeting at joint O are:

For member OA

$$M_{OA} = 2\,E\,K_{OA}\left(-\,2\,\theta_0 - \theta_A + \frac{3\,\delta_{OA}}{l_{OA}}\right) \qquad (3a)$$

$$M_{AO} = 2\,E\,K_{OA}\left(-2\,\theta_A - \theta_0 + \frac{3\,\delta_{OA}}{l_{OA}}\right) \quad \ldots\ldots (3b)$$

and for member OB

$$M_{OB} = 2\,E\,K_{OB}\left(-2\,\theta_0 - \theta_B + \frac{3\,\delta_{OB}}{l_{OB}}\right) \quad \ldots\ldots (4a)$$

$$M_{BO} = 2\,E\,K_{OB}\left(-2\,\theta_B - \theta_0 + \frac{3\,\delta_{OB}}{l_{OB}}\right) \quad \ldots\ldots (4b)$$

From Eqs. 1 and 3

$$M_{OA} = -4\,E\,K_{OA}\,\theta_0 \quad \ldots\ldots (5a)$$

$$M_{AO} = -2\,E\,K_{OA}\,\theta_0 \quad \ldots\ldots (5b)$$

from Eqs. 2 and 4

$$M_{OB} = \frac{-\theta_0}{\frac{1}{3\,E\,K_{OB}} + \frac{1}{c\,l^2}} \quad \ldots\ldots (6a)$$

$$M_{BO} = 0 \quad \ldots\ldots (6b)$$

and from Eqs. 5 and 6 the moment-distribution factors and the carry-over factors are as follows:

$$M_{OA}:M_{OB} = 4\,E\,K_{OA}:\frac{1}{\frac{1}{3\,E\,K_{OB}} + \frac{1}{c\,l^2}} \quad \ldots\ldots (7a)$$

$$M_{AO}:M_{OA} = 1:2 \quad \ldots\ldots (7b)$$

and

$$M_{BO}:M_{OB} = 0 \quad \ldots\ldots (7c)$$

The applied moment M_0 can be balanced by a single cycle of moment distribution:

$$M_{OA} = \frac{4\,E\,K_{OA}}{4\,E\,K_{OA} + \frac{1}{\frac{1}{3\,E\,K_{OB}} + \frac{1}{c\,l^2}}}\,M_0 \quad \ldots\ldots (8a)$$

$$M_{OB} = \frac{\frac{1}{\frac{1}{3\,E\,K_{OB}} + \frac{1}{c\,l^2}}}{4\,E\,K_{OA} + \frac{1}{\frac{1}{3\,E\,K_{OB}} + \frac{1}{c\,l^2}}}\,M_0 \quad \ldots\ldots (8b)$$

$$M_{AO} = \tfrac{1}{2}\,M_{OA} \quad \ldots\ldots (8c)$$

and

$$M_{BO} = 0 \quad \ldots\ldots (8d)$$

Principle Defining Moment Distribution.—The sets of two equations defining the degree of fixity of each member are called the boundary equations because

they describe the end conditions of the member. These two equations must be independent of each other, independent of statics, and also, independent of the two slope-deflection equations. From the previous example, the following principle is clear: Any two arbitrary, linear, independent, homogeneous boundary equations for each member will define a system or process of moment distribution and vice versa. Any two linear, independent, homogeneous boundary equations will yield a formal process of moment distribution; however, many of these processes may not converge at all and many more will converge slowly. Evidently only those that converge reasonably rapidly can be used. In actual practice, the boundary equations are not chosen; these are often dictated by the geometry of the structure to be analyzed. To cite a few examples from among the infinite number of possible systems of moment distribution, Eqs. 2 are replaced by Eqs. 9 and 10. These equations are applicable to Fig. 9, in which the joint B is displaced vertically without rotation. For Fig. 9,

$$\theta_B = 0 \quad \ldots\ldots (9)$$

and

$$M_{BO} : M_{OB} = r \quad \ldots\ldots (10)$$

Eqs. 1, 9, and 10 define a process of moment distribution having the following characteristics:

$$M_{OA} : M_{OB} = K_{OA} : K_{OB}\, v_{OB} \quad \ldots\ldots (11a)$$

$$M_{AO} : M_{OA} = \tfrac{1}{2} \quad \ldots\ldots (11b)$$

and

$$M_{BO} : M_{OB} = r \quad \ldots\ldots (11c)$$

in which

$$v_{OB} = \frac{1}{2 - 2r} \quad \ldots\ldots (12)$$

Here θ_B is zero while δ_{OB} is not zero. Eq. 12 can be compared with $v = \dfrac{3}{4 - 2r}$ which applies to the case when $\delta_{OB} = 0$ while θ_B is not zero. It can be seen that v is a correction factor such that Kv represents the relative resistance to end rotation.

Cantilever Moment Distribution.—For the special case in which $r = -1$, Eq. 12 results in $v_{OB} = \frac{1}{4}$, and member OB takes no shear during moment distribution. Member OB acts similar to a cantilever fixed at B and loaded at O by the moment M_{OB}. For this reason this special process of moment distribution has been called "cantilever moment distribution." This method's unique property of producing no shear change during moment distribution makes it an excellent device for solving certain problems involving sidesway.

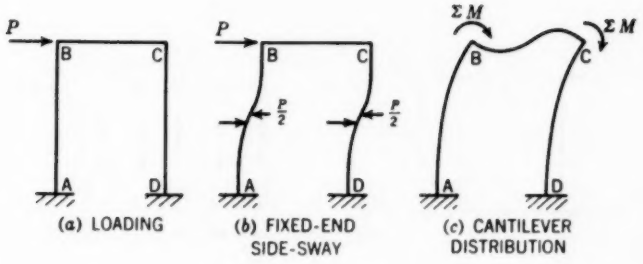

FIG. 1.—A SIMPLE BENT

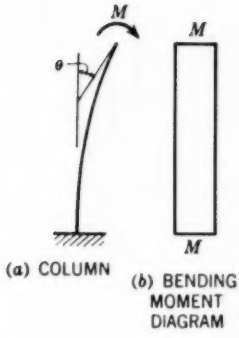

FIG. 2.—CANTILEVER MOMENTS

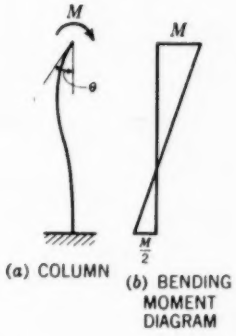

FIG. 3.—ROTATION WITHOUT TRANSLATION

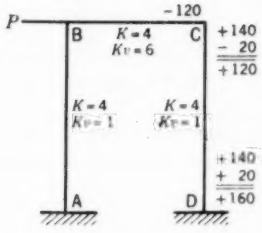

FIG. 4.—ANALYSIS OF A SIMPLE BENT

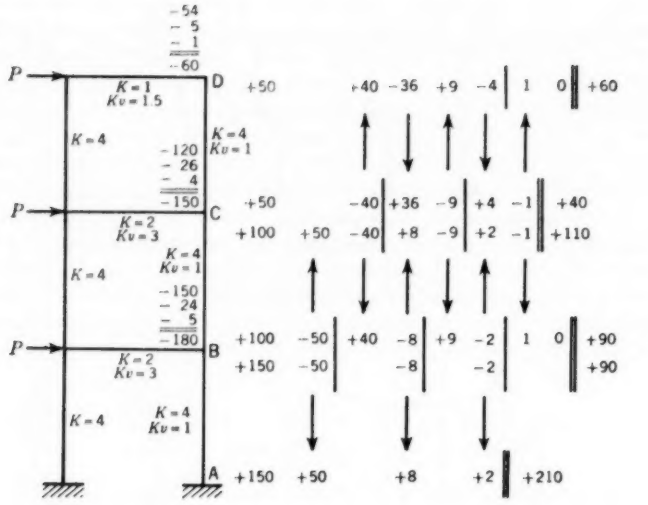

FIG. 5.—ANALYSIS OF A THREE-STORY BENT

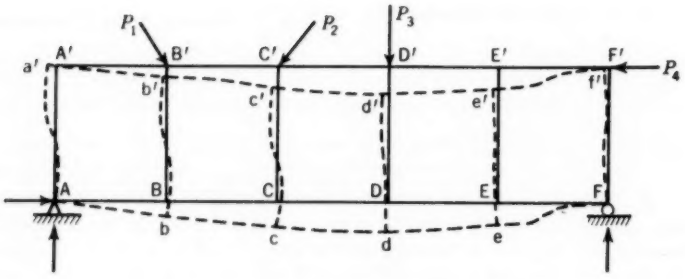

FIG. 6.—A PARALLEL CHORD VIERENDEEL TRUSS

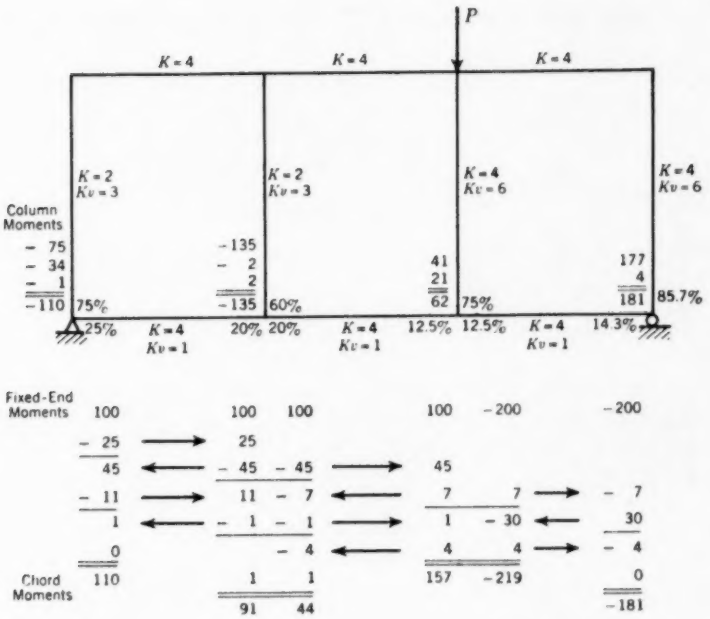

FIG. 7.—AN UNSYMMETRICAL VIERENDEEL TRUSS

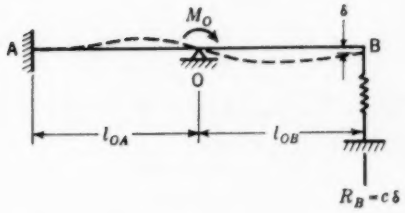

FIG. 8.—A CONTINUOUS GIRDER WITH ONE END ON AN ELASTIC SUPPORT

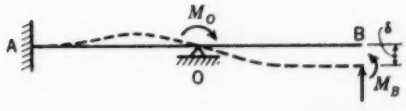

FIG. 9.—A CONTINUOUS GIRDER WITH ONE END DISPLACED VERTICALLY WITHOUT ROTATION

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